

## 9.11 Triple Integral

Functions of three variables:  $f(x, y, z)$ ,  $g(x, y, z)$ , ...

Triple integrals:  $\iiint_G f(x, y, z) dV$ ,  $\iiint_G g(x, y, z) dV$ , ...

Riemann sum:  $\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p f(u_i, v_j, w_k) \Delta x_i \Delta y_j \Delta z_k$

Small changes:  $\Delta x_i$ ,  $\Delta y_j$ ,  $\Delta z_k$

Limits of integration:  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $r$ ,  $s$

Regions of integration:  $G$ ,  $T$ ,  $S$

Cylindrical coordinates:  $r$ ,  $\theta$ ,  $z$

Spherical coordinates:  $r$ ,  $\theta$ ,  $\phi$

Volume of a solid:  $V$

Mass of a solid:  $m$

Density:  $\mu(x, y, z)$

Coordinates of center of mass:  $\bar{x}$ ,  $\bar{y}$ ,  $\bar{z}$

First moments:  $M_{xy}$ ,  $M_{yz}$ ,  $M_{xz}$

Moments of inertia:  $I_{xy}$ ,  $I_{yz}$ ,  $I_{xz}$ ,  $I_x$ ,  $I_y$ ,  $I_z$ ,  $I_0$

### 1099. Definition of Triple Integral

The triple integral over a parallelepiped  $[a, b] \times [c, d] \times [r, s]$  is defined to be

$$\iiint_{[a, b] \times [c, d] \times [r, s]} f(x, y, z) dV = \lim_{\substack{\max \Delta x_i \rightarrow 0 \\ \max \Delta y_j \rightarrow 0 \\ \max \Delta z_k \rightarrow 0}} \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p f(u_i, v_j, w_k) \Delta x_i \Delta y_j \Delta z_k,$$

where  $(u_i, v_j, w_k)$  is some point in the parallelepiped

$(x_{i-1}, x_i) \times (y_{j-1}, y_j) \times (z_{k-1}, z_k)$ , and  $\Delta x_i = x_i - x_{i-1}$ ,

$\Delta y_j = y_j - y_{j-1}$ ,  $\Delta z_k = z_k - z_{k-1}$ .

$$1100. \iiint_G [f(x, y, z) + g(x, y, z)] dV = \iiint_G f(x, y, z) dV + \iiint_G g(x, y, z) dV$$



$$1101. \iiint_G [f(x, y, z) - g(x, y, z)] dV = \iiint_G f(x, y, z) dV - \iiint_G g(x, y, z) dV$$

$$1102. \iiint_G kf(x, y, z) dV = k \iiint_G f(x, y, z) dV,$$

where  $k$  is a constant.

1103. If  $f(x, y, z) \geq 0$  and  $G$  and  $T$  are nonoverlapping basic regions, then

$$\iiint_{G \cup T} f(x, y, z) dV = \iiint_G f(x, y, z) dV + \iiint_T f(x, y, z) dV.$$

Here  $G \cup T$  is the union of the regions  $G$  and  $T$ .

1104. Evaluation of Triple Integrals by Repeated Integrals

If the solid  $G$  is the set of points  $(x, y, z)$  such that  $(x, y) \in R$ ,  $\chi_1(x, y) \leq z \leq \chi_2(x, y)$ , then

$$\iiint_G f(x, y, z) dx dy dz = \iint_R \left[ \int_{\chi_1(x, y)}^{\chi_2(x, y)} f(x, y, z) dz \right] dx dy,$$

where  $R$  is projection of  $G$  onto the  $xy$ -plane.

If the solid  $G$  is the set of points  $(x, y, z)$  such that  $a \leq x \leq b$ ,  $\varphi_1(x) \leq y \leq \varphi_2(x)$ ,  $\chi_1(x, y) \leq z \leq \chi_2(x, y)$ , then

$$\iiint_G f(x, y, z) dx dy dz = \int_a^b \left[ \int_{\varphi_1(x)}^{\varphi_2(x)} \left( \int_{\chi_1(x, y)}^{\chi_2(x, y)} f(x, y, z) dz \right) dy \right] dx$$

1105. Triple Integrals over Parallelepiped

If  $G$  is a parallelepiped  $[a, b] \times [c, d] \times [r, s]$ , then

$$\iiint_G f(x, y, z) dx dy dz = \int_a^b \left[ \int_c^d \left( \int_r^s f(x, y, z) dz \right) dy \right] dx.$$

In the special case where the integrand  $f(x, y, z)$  can be written as  $g(x)h(y)k(z)$  we have

$$\iiint_G f(x, y, z) dx dy dz = \left( \int_a^b g(x) dx \right) \left( \int_c^d h(y) dy \right) \left( \int_r^s k(z) dz \right).$$

### 1106. Change of Variables

$$\iiint_G f(x, y, z) dx dy dz =$$

$$= \iiint_S f[x(u, v, w), y(u, v, w), z(u, v, w)] \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dx dy dz,$$

where  $\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$  is the **jacobian** of

the transformations  $(x, y, z) \rightarrow (u, v, w)$ , and  $S$  is the pull-back of  $G$  which can be computed by  $x = x(u, v, w)$ ,  
 $y = y(u, v, w)$   
 $z = z(u, v, w)$  into the definition of  $G$ .

### 1107. Triple Integrals in Cylindrical Coordinates

The differential  $dx dy dz$  for cylindrical coordinates is

$$dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| dr d\theta dz = r dr d\theta dz.$$

Let the solid  $G$  is determined as follows:

$$(x, y) \in R, \chi_1(x, y) \leq z \leq \chi_2(x, y),$$

where  $R$  is projection of  $G$  onto the  $xy$ -plane. Then

$$\iiint_G f(x, y, z) dx dy dz = \iiint_S f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

$$= \iint_{R(r,\theta)} \left[ \int_{\chi_1(r \cos \theta, r \sin \theta)}^{\chi_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) dz \right] r dr d\theta.$$

Here  $S$  is the pullback of  $G$  in cylindrical coordinates.

### 1108. Triple Integrals in Spherical Coordinates

The Differential  $dx dy dz$  for Spherical Coordinates is

$$dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} \right| dr d\theta d\varphi = r^2 \sin \theta dr d\theta d\varphi$$

$$\iiint_G f(x, y, z) dx dy dz =$$

$$= \iiint_S f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) r^2 \sin \theta dr d\theta d\varphi,$$

where the solid  $S$  is the pullback of  $G$  in spherical coordinates. The angle  $\theta$  ranges from  $0$  to  $2\pi$ , the angle  $\varphi$  ranges from  $0$  to  $\pi$ .

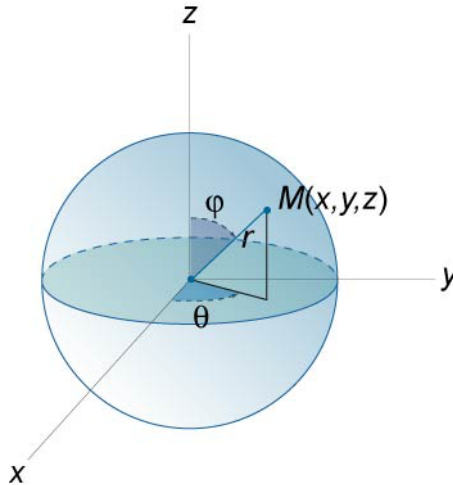


Figure 202.

**1109. Volume of a Solid**

$$V = \iiint_G dx dy dz$$

**1110. Volume in Cylindrical Coordinates**

$$V = \iiint_{S(r,\theta,z)} r dr d\theta dz$$

**1111. Volume in Spherical Coordinates**

$$V = \iiint_{S(r,\theta,\phi)} r^2 \sin \theta dr d\theta d\phi$$

**1112. Mass of a Solid**

$$m = \iiint_G \mu(x,y,z) dV,$$

where the solid occupies a region  $G$  and its density at a point  $(x,y,z)$  is  $\mu(x,y,z)$ .

**1113. Center of Mass of a Solid**

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \iiint_G x \mu(x,y,z) dV,$$

$$M_{xz} = \iiint_G y \mu(x,y,z) dV,$$

$$M_{xy} = \iiint_G z \mu(x,y,z) dV$$

are the first moments about the coordinate planes  $x=0$ ,  $y=0$ ,  $z=0$ , respectively,  $\mu(x,y,z)$  is the density function.

**1114. Moments of Inertia about the  $xy$ -plane (or  $z=0$ ),  $yz$ -plane ( $x=0$ ), and  $xz$ -plane ( $y=0$ )**

$$I_{xy} = \iiint_G z^2 \mu(x, y, z) dV,$$

$$I_{yz} = \iiint_G x^2 \mu(x, y, z) dV,$$

$$I_{xz} = \iiint_G y^2 \mu(x, y, z) dV.$$

**1115. Moments of Inertia about the x-axis, y-axis, and z-axis**

$$I_x = I_{xy} + I_{xz} = \iiint_G (z^2 + y^2) \mu(x, y, z) dV,$$

$$I_y = I_{xy} + I_{yz} = \iiint_G (z^2 + x^2) \mu(x, y, z) dV,$$

$$I_z = I_{xz} + I_{yz} = \iiint_G (y^2 + x^2) \mu(x, y, z) dV.$$

**1116. Polar Moment of Inertia**

$$I_0 = I_{xy} + I_{yz} + I_{xz} = \iiint_G (x^2 + y^2 + z^2) \mu(x, y, z) dV$$

